Lecture 35

Spectral Expansions of Source Fields—Sommerfeld Integrals

In previous lectures, we have assumed plane waves in finding closed form solutions. Plane waves are simple waves, and their reflections off a flat surface or a planarly layered medium can be found easily. But plane waves are mathematical idealizations that are not encountered in the real world.

When we have a source like a point source, it generates a spherical wave. We do not know how to reflect exactly a spherical wave off a planar interface. But by expanding a spherical wave in terms of sum of plane waves and evanescennt waves using Fourier transform technique, we can solve for the solution of a point source over a layered medium easily in terms of spectral integrals using Fourier transform in space. Sommerfeld was the first person to have done this, and hence, these integrals are often called Sommerfeld integrals.

Finally, we shall apply the method of stationary phase to approximate these integrals to elucidate their physics. From this, we can see ray physics and Fermat's principle theory emerging from the complicated mathematics. It reminds us of a lyric from the musical The Sound of Music—Ray, a drop of golden sun! Ray has mesmerized the human mind, and it will be interesting to see if the mathematics behind it is equally enchanting.

By this time, you probably feel inundated by the ocean of knowledge that you are imbibing. But if you can assimilate them, it will be an exhilarating experience.

35.1 Spectral Representations of Sources

As mentioned above, a plane wave is a mathematical idealization that does not exist in the real world. In practice, waves are nonplanar in nature as they are generated by finite sources, such as antennas and scatterers: For example, a point source generates a spherical wave which is nonplanar. Fortunately, these non-planar waves can be expanded in terms of sum of plane waves. Once this is done, then the study of non-plane-wave reflections from a layered medium becomes routine.

In the following, we shall show how waves resulting from a point source can be expanded in terms of plane waves summation. This topic is found in many textbooks [1,33,36,107,108, 196,220,230].

35.1.1 A Point Source—Fourier Expansion and Contour Integration

There are a number of ways to derive the plane wave expansion of a point source. We will illustrate one of the ways. The Fourier expansion in space, or spectral decomposition, or the plane-wave expansion of the field due to a point source could be derived using Fourier transform technique. First, notice that the scalar wave equation with a point source at the origin is

$$\left(\nabla^2 + k_0^2\right)\phi(x,y,z) = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k_0^2\right]\phi(x,y,z) = -\delta(x)\,\delta(y)\,\delta(z). \tag{35.1.1}$$

The above equation could then be solved in the spherical coordinates, yielding the solution given in the previous lecture, namely, Green's function with the source point at the origin, or[1]

$$\phi(x,y,z) = \phi(r) = \frac{e^{ik_0 r}}{4\pi r}. \tag{35.1.2}$$

The solution is entirely spherically symmetric due to the symmetry and location of the point source.

Next, assuming that the Fourier transform of $\phi(x,y,z)$ exists,[2] we can write

$$\phi(x,y,z) = \frac{1}{(2\pi)^3}\iiint\limits_{-\infty}^{\infty} dk_x dk_y dk_z\, \tilde{\phi}(k_x,k_y,k_z) e^{ik_x x + ik_y y + ik_z z}. \tag{35.1.3}$$

Then we substitute the above into (35.1.1), after exchanging the order of differentiation and integration,[3] one can simplify the Laplacian operator in the Fourier space, or spectral domain, to arrive at

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = -k_x^2 - k_y^2 - k_z^2$$

Then, together with the Fourier representation of the delta function, which is[4]

$$\delta(x)\,\delta(y)\,\delta(z) = \frac{1}{(2\pi)^3}\iiint\limits_{-\infty}^{\infty} dk_x dk_y dk_z\, e^{ik_x x + ik_y y + ik_z z} \tag{35.1.4}$$

[1]From this point onward, we will adopt the $\exp(-i\omega t)$ time convention to be commensurate with the optics and physics literatures.

[2]The Fourier transform of a function $f(x)$ exists if it is absolutely integrable, namely that $\int_{-\infty}^{\infty} |f(x)|dx$ is finite (see [122]).

[3]Exchanging the order of differentiation and integration is allowed if the integral converges after the exchange.

[4]We have made use of that $\delta(x) = 1/(2\pi)\int_{-\infty}^{\infty} dk_x \exp(ik_x x)$ three times.

we convert (35.1.1) into

$$\iiint_{-\infty}^{\infty} dk_x dk_y dk_z\,[k_0^2 - k_x^2 - k_y^2 - k_z^2]\tilde{\phi}(k_x, k_y, k_z)e^{ik_x x + ik_y y + ik_z z} \tag{35.1.5}$$

$$= -\iiint_{-\infty}^{\infty} dk_x dk_y dk_z\, e^{ik_x x + ik_y y + ik_z z}. \tag{35.1.6}$$

Since the above is equal for all x, y, and z, we can Fourier inverse transform the above to get

$$\tilde{\phi}(k_x, k_y, k_z) = \frac{-1}{k_0^2 - k_x^2 - k_y^2 - k_z^2}. \tag{35.1.7}$$

Consequently, using this in (35.1.3), we have

$$\phi(x, y, z) = \frac{-1}{(2\pi)^3}\iiint_{-\infty}^{\infty} d\mathbf{k}\, \frac{e^{ik_x x + ik_y y + ik_z z}}{k_0^2 - k_x^2 - k_y^2 - k_z^2}. \tag{35.1.8}$$

where $d\mathbf{k} = dk_x dk_y dk_z$. The above expresses the fact the $\phi(x, y, z)$ which is a spherical wave by (35.1.2), is expressed as an integral summation of "plane waves". But these "plane waves" are not physical plane waves in free space since $k_x^2 + k_y^2 + k_z^2 \neq k_0^2$. In other words, the "plane waves" do not satisfy the dispersion relation of a physical plane wave.

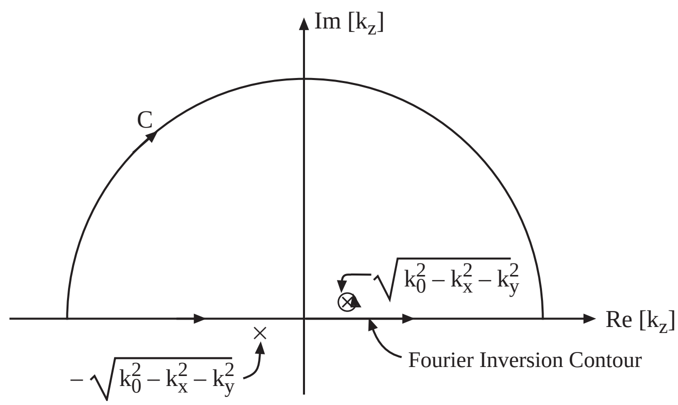

Figure 35.1: The integration along the real axis is equal to the integration along C plus the residue of the pole at $(k_0^2 - k_x^2 - k_y^2)^{1/2}$, by invoking Jordan's lemma.

Weyl Identity–Plane-Wave Expansion of a Point-Source Field

To make the plane waves in (35.1.8) into physical plane waves, we have to massage it into a different form. We rearrage the integrals in (35.1.8) so that the dk_z integral is performed first. In other words,

$$\phi(\mathbf{r}) = \frac{1}{(2\pi)^3} \iint\limits_{-\infty}^{\infty} dk_x dk_y e^{ik_x x + ik_y y} \int_{-\infty}^{\infty} dk_z \frac{e^{ik_z z}}{k_z^2 - (k_0^2 - k_x^2 - k_y^2)} \tag{35.1.9}$$

where we have deliberately rearrange the denominator with k_z being the variable in the inner integral. Then the integrand has poles at $k_z = \pm(k_0^2 - k_x^2 - k_y^2)^{1/2}$.[5] Moreover, for real k_0, and real values of k_x and k_y, these two poles lie on the real axis, rendering the integral in (35.1.8) undefined. However, if a small loss is assumed in k_0 such that $k_0 = k_0' + ik_0''$, then the poles are off the real axis (see Figure 35.1), and the integrals in (35.1.8) are well-defined. In actual fact, this is intimately related to the uniqueness principle we have studied before: An infinitesimal loss is needed to guarantee uniqueness in an open space as shall be explained below.

First, the reason is that without loss, $|\phi(\mathbf{r})| \sim O(1/r)$, $r \to \infty$ is not strictly absolutely integrable, and hence, its Fourier transform does not exist [53]: The manipulation that leads to (35.1.8) is not strictly correct. Second, the introduction of a small loss also guarantees the radiation condition and the uniqueness of the solution to (35.1.1), and therefore, the equality of (35.1.2) and (35.1.8) [36].

Observe that in (35.1.8), when $z > 0$, the integrand is exponentially small when $\Im m[k_z] \to \infty$. Therefore, by Jordan's lemma [92], the integration for k_z over the contour C as shown in Figure 35.1 vanishes. Then, by Cauchy's theorem [92], the integration over the Fourier inversion contour on the real axis is the same as integrating over the pole singularity located at $(k_0^2 - k_x^2 - k_y^2)^{1/2}$, yielding the residue of the pole (see Figure 35.1). Consequently, after doing the residue evaluation, we have

$$\phi(x, y, z) = \frac{i}{2(2\pi)^2} \iint\limits_{-\infty}^{\infty} dk_x dk_y \frac{e^{ik_x x + ik_y y + ik_z' z}}{k_z'}, \quad z > 0, \tag{35.1.10}$$

where $k_z' = (k_0^2 - k_x^2 - k_y^2)^{1/2}$ is the value of k_z at the pole location.

Similarly, for $z < 0$, we can add a contour C in the lower-half plane that contributes zero to the integral, one can deform the contour to pick up the pole contribution. Therefore, the integral is equal to the pole contribution at $k_z' = -(k_0^2 - k_x^2 - k_y^2)^{1/2}$ (see Figure 35.1). As such, the result valid for all z can be written as

$$\phi(x, y, z) = \frac{i}{2(2\pi)^2} \iint\limits_{-\infty}^{\infty} dk_x dk_y \frac{e^{ik_x x + ik_y y + ik_z' |z|}}{k_z'}, \quad \text{all } z. \tag{35.1.11}$$

By the uniqueness of the solution to the partial differential equation (35.1.1) satisfying radiation condition at infinity, we can equate (35.1.2) and (35.1.11), yielding the identity

$$\frac{e^{ik_0 r}}{r} = \frac{i}{2\pi} \iint\limits_{-\infty}^{\infty} dk_x dk_y \frac{e^{ik_x x + ik_y y + ik_z |z|}}{k_z}, \tag{35.1.12}$$

[5] In (35.1.8), the pole is located at $k_x^2 + k_y^2 + k_z^2 = k_0^2$. This equation describes a sphere in $\mathbf{k}$ space, known as the Ewald's sphere [231].

where $k_x^2 + k_y^2 + k_z^2 = k_0^2$, or $k_z = (k_0^2 - k_x^2 - k_y^2)^{1/2}$. The above is known as the *Weyl identity* (Weyl 1919). To ensure the radiation condition, we require that $\Im m[k_z] > 0$ and $\Re e[k_z] > 0$ over all values of k_x and k_y in the integration. Furthermore, Equation (35.1.12) could be interpreted as an integral summation of plane waves propagating in all directions, including evanescent waves. It is the plane-wave expansion (including evanescent wave) of a spherical wave.

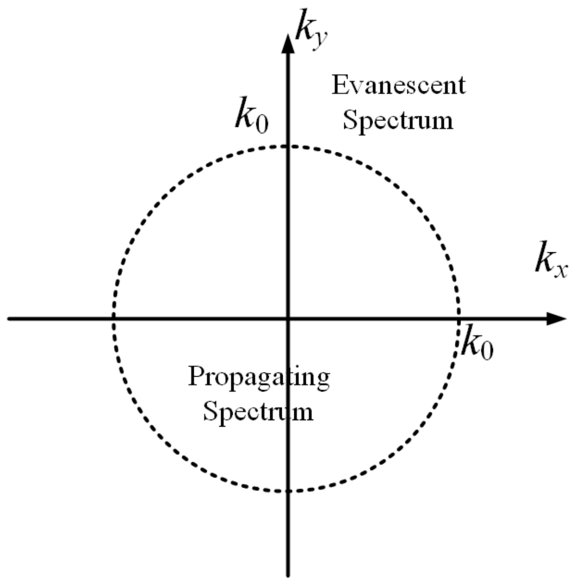

Figure 35.2: The integral in the Weyl identity is done over the entire k_x and k_y plane. The wave is propagating for $\mathbf{k}_\rho = \hat{x}k_x + \hat{y}k_y$ vectors inside the disk, while the wave is evanescent for $\mathbf{k}_\rho$ outside the disk.

One can also interpret the above as a 2D surface integral in the Fourier space over the k_x and k_y plane or variables. When $k_x^2 + k_y^2 < k_0^2$, or the spatial spectrum involving k_x and k_y is inside a disk of radius k_0, the waves are propagating waves. But for contributions outside this disk, the waves are evanescent (see Figure 35.2). And the high Fourier (or spectral) components of the Fourier spectrum correspond to evanescent waves. The high spectral components, which are related to the evanescent waves, are important for reconstructing the singularity of the Green's function.[6]

[6]It may be difficult to wrap your head around so many new concepts, and you will have to contemplate on them to deeply understand them.

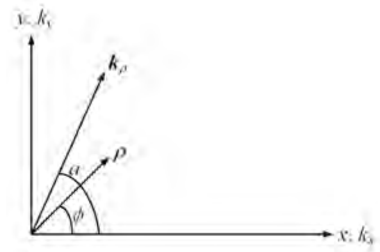

Figure 35.3: The $\mathbf{k}_\rho$ and the $\boldsymbol{\rho}$ vectors on the $k_x k_y$ plane and the xy plane. The two planes are superposed.

Sommerfeld Identity—A Semi-Infinite Integral

The Weyl identity has double integral, and hence, is more difficult to integrate numerically. Here, we shall derive the Sommerfeld identity which has only one semi-infinite integral. First, in (35.1.12), we express the integral in cylindrical coordinates and write $\mathbf{k}_\rho = \hat{x}k_\rho \cos\alpha + \hat{y}k_\rho \sin\alpha$, $\boldsymbol{\rho} = \hat{x}\rho\cos\phi + \hat{y}\rho\sin\phi$ (see Figure 35.3), and $dk_x dk_y = k_\rho dk_\rho \, d\alpha$. Then, $k_x x + k_y y = \mathbf{k}_\rho \cdot \boldsymbol{\rho} = k_\rho \cos(\alpha - \phi)$, and with the appropriate change of variables, we have

$$\frac{e^{ik_0 r}}{r} = \frac{i}{2\pi} \int_0^\infty k_\rho dk_\rho \int_0^{2\pi} d\alpha \frac{e^{ik_\rho \rho \cos(\alpha-\phi)+ik_z|z|}}{k_z}, \tag{35.1.13}$$

where $k_z = (k_0^2 - k_x^2 - k_y^2)^{1/2} = (k_0^2 - k_\rho^2)^{1/2}$, where in cylindrical coordinates, in the $\mathbf{k}_\rho$-space, or the Fourier space, $k_\rho^2 = k_x^2 + k_y^2$. Then, using the integral identity for Bessel functions given by[7]

$$J_0(k_\rho \rho) = \frac{1}{2\pi} \int_0^{2\pi} d\alpha \, e^{ik_\rho \rho \cos(\alpha - \phi)}, \tag{35.1.14}$$

(35.1.13) becomes

$$\frac{e^{ik_0 r}}{r} = i \int_0^\infty dk_\rho \frac{k_\rho}{k_z} J_0(k_\rho \rho) e^{ik_z|z|}. \tag{35.1.15}$$

The above is also known as the *Sommerfeld identity* (Sommerfeld 1909 [121]; [220][p. 242]). Its physical interpretation is that a spherical wave can now be expanded as an integral summation of conical waves or cylindrical waves in the ρ direction, times a plane wave in the z direction over all wave numbers k_ρ. This wave is evanescent in the $\pm z$ direction when $k_\rho > k_0$ as shown in Figure 35.2.

[7]See Chew [36], or Whitaker and Watson(1927) [232].

By using the fact that $J_0(k_\rho\rho) = 1/2[H_0^{(1)}(k_\rho\rho) + H_0^{(2)}(k_\rho\rho)]$, and the reflection formula that $H_0^{(1)}(e^{i\pi}x) = -H_0^{(2)}(x)$, a variation of the above identity can be derived as [36]

$$\frac{e^{ik_0 r}}{r} = \frac{i}{2}\int_{-\infty}^{\infty} dk_\rho\, \frac{k_\rho}{k_z} H_0^{(1)}(k_\rho\rho) e^{ik_z|z|}. \tag{35.1.16}$$

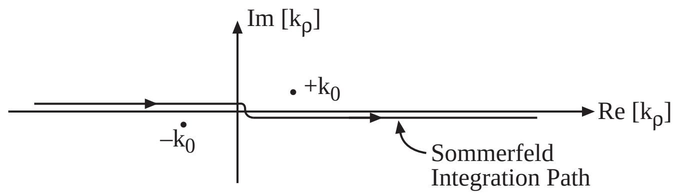

Figure 35.4: Sommerfeld integration path.

Since $H_0^{(1)}(x)$ has a logarithmic branch-point singularity at $x = 0$,[8] and $k_z = (k_0^2 - k_\rho^2)^{1/2}$ has algebraic branch-point singularities at $k_\rho = \pm k_0$, the integral in Equation (35.1.16) is undefined unless we stipulate also the path of integration. Thus, a path of integration adopted by Sommerfeld, which is even good for a lossless medium, is shown in Figure 35.4. Because of the manner in which we have selected the reflection formula for Hankel functions, i.e., $H_0^{(1)}(e^{i\pi}x) = -H_0^{(2)}(x)$, the path of integration should be above the logarithmic branch-point singularity at the origin. With this definition of the Sommerfeld integration, the integral is well defined even when there is no loss, i.e., when the branch points $\pm k_0$ are on the real axis.

35.2 A Source on Top of a Layered Medium

Previously, we have studied the propagation of plane electromagnetic waves from a single dielectric interface in Section 14.1 as well as through a layered medium in Section 16.1. It can be shown that plane waves reflecting from a layered medium can be decomposed into TE-type plane waves, where $E_z = 0$, $H_z \neq 0$, and TM-type plane waves, where $H_z = 0$, $E_z \neq 0$.[9] One also sees how the field due to a point source can be expanded into plane waves in Section 35.1.

In view of the above observations, when a point source is on top of a layered medium, it is then best to decompose its field in terms of plane waves of TE-type and TM-type. Then, the nonzero component of E_z characterizes TM-to-z waves, while the nonzero component of H_z characterizes TE-to-z waves. Hence, given a field, its TM and TE components can be extracted readily. Furthermore, if these TM and TE components are expanded in terms of plane waves, their propagations in a layered medium can be studied easily.

The problem of a vertical electric dipole on top of a half space was first solved by Sommerfeld (1909) [121] using Hertzian potentials, which are related to the z components of the

[8] $H_0^{(1)}(x) \sim \frac{2i}{\pi}\ln(x)$, see Chew [36][p. 14], or Abromawitz or Stegun [129].
[9] *Chew, Waves and Fields in Inhomogeneous Media* [36]; *Kong, Electromagnetic Wave Theory* [33].

electromagnetic field. The work is later generalized to layered media, as discussed in the literature. Later, Kong (1972) [233] suggested the use of the z components of the electromagnetic field instead of the Hertzian potentials.

35.2.1 Electric Dipole Fields–Spectral Expansion

The representation of a spherical wave in terms of plane waves can be done using Weyl identity or Sommerfeld identiy. Here, we will use Sommerfeld identity in anticipation of simpler numerical integration, since only single integrals are involved. The $\mathbf{E}$ field in a homogeneous medium due to a point current source or a Hertzian dipole directed in the $\hat{\alpha}$ direction, $\mathbf{J} = \hat{\alpha} I\ell\, \delta(\mathbf{r})$, is derivable via the vector potential method or the dyadic Green's function approach. Then, using the dyadic Green's function approach, or the vector/scalar potential approach, the field due to a Hertzian dipole is given by

$$\mathbf{E}(\mathbf{r}) = i\omega\mu \left(\bar{\mathbf{I}} + \frac{\nabla\nabla}{k^2} \right) \cdot \hat{\alpha} I\ell\, \frac{e^{ikr}}{4\pi r}, \tag{35.2.1}$$

where $I\ell$ is the current moment and $k = \omega\sqrt{\mu\epsilon}$, the wave number of the homogeneous medium. Furthermore, from $\nabla \times \mathbf{E} = i\omega\mu\mathbf{H}$, the magnetic field due to a Hertzian dipole is shown to be given by

$$\mathbf{H}(\mathbf{r}) = \nabla \times \hat{\alpha} I\ell\, \frac{e^{ikr}}{4\pi r}. \tag{35.2.2}$$

With the above fields, their TM-to-z and TE-to-z components can be extracted easily in anticipation of their plane wave expansions for propagation through layered media.

(a) Vertical Electric Dipole (VED)—Spectral Expansion

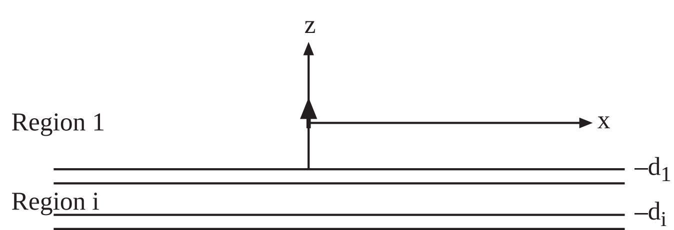

Figure 35.5: A vertical electric dipole over a layered medium.

A vertical electric dipole shown in Figure 35.5 has $\hat{\alpha} = \hat{z}$; hence, in anticipation of their plane wave expansions, the TM-to-z component of the field is characterized by $E_z \neq 0$ or that

$$E_z = \frac{i\omega\mu I\ell}{4\pi k^2} \left(k^2 + \frac{\partial^2}{\partial z^2} \right) \frac{e^{ikr}}{r}, \tag{35.2.3}$$

and the TE component of the field is characterized by

$$H_z = 0, \tag{35.2.4}$$

implying the absence of the TE-to-z field.

Next, using the Sommerfeld identity (35.1.16) in the above, and after exchanging the order of integration and differentiation, we have[10]

$$E_z = \frac{-I\ell}{4\pi\omega\epsilon} \int_0^\infty dk_\rho \frac{k_\rho^3}{k_z} J_0(k_\rho \rho) e^{ik_z|z|},\ |z| \neq 0 \tag{35.2.5}$$

after noting that $k_\rho^2 + k_z^2 = k^2$. Notice that now Equation (35.2.5) expands the z component of the electric field in terms of cylindrical waves in the ρ direction and a plane wave in the z direction. (Cylindrical waves actually are linear superpositions of plane waves, because we can work backward from (35.1.16) to (35.1.12) to see this.) As such, the integrand in (35.2.5) in fact consists of a linear superposition of TM-type plane waves. The above is also the *primary field* generated by the source.[11]

Consequently, for a VED on top of a stratified medium as shown, expanding the source field in terms of plane waves, the downgoing plane waves from the point source will be reflected like TM waves with the generalized reflection coefficient $\tilde{R}_{12}^{TM}$. Hencef, over a stratified medium, the field in region 1 can be written as

$$E_{1z} = \frac{-I\ell}{4\pi\omega\epsilon_1} \int_0^\infty dk_\rho \frac{k_\rho^3}{k_{1z}} J_0(k_\rho \rho) \left[e^{ik_{1z}|z|} + \tilde{R}_{12}^{TM} e^{ik_{1z}z + 2ik_{1z}d_1} \right], \tag{35.2.6}$$

where $k_{1z} = (k_1^2 - k_\rho^2)^{\frac{1}{2}}$, and $k_1^2 = \omega^2 \mu_1 \epsilon_1$, the wave number in region 1.

The phase-matching condition dictates that the transverse variation of the field in all the regions must be the same. Consequently, in the i-th region, the solution becomes[12]

$$\epsilon_i E_{iz} = \frac{-I\ell}{4\pi\omega} \int_0^\infty dk_\rho \frac{k_\rho^3}{k_{1z}} J_0(k_\rho \rho) A_i \left[e^{-ik_{iz}z} + \tilde{R}_{i,i+1}^{TM} e^{ik_{iz}z + 2ik_{iz}d_i} \right]. \tag{35.2.7}$$

Notice that Equation (35.2.7) is now expressed in terms of $\epsilon_i E_{iz}$ because $\epsilon_i E_{iz}$ reflects and transmits like H_{iy}, the transverse component of the magnetic field or TM waves.[13] Therefore, $\tilde{R}_{i,i+1}^{TM}$ and A_i could be obtained using the methods discussed in *Chew, Waves and Fields in Inhomogeneous Media* [122].

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[10] By using (35.1.16) in (35.2.3), the $\partial^2/\partial z^2$ operating on $e^{ik_z|z|}$ produces a Dirac delta function singularity. But in (35.2.5). we ignore the delta function since $|z| \neq 0$. Detail discussion on this can be found in the chapter on dyadic Green's function in *Chew, Waves and Fields in Inhomogeneous Media* [36].

[11] One can perform a sanity check on the odd and even symmetry of the fields' z-component by sketching the fields of a static horizontal electric dipole.

[12] It will take quite a bit of work to get this expression, but you just need to know that it can be done, and know where to look for the resources for it.

[13] See *Chew, Waves and Fields in Inhomogeneous Media* [36], p. 46, (2.1.6) and (2.1.7). Or we can gather from (14.1.6) to (14.1.7) that the $\mu_i H_{iz}$ transmits like E_{iy} at a dielectric interface, and by duality, $\epsilon_i E_{iz}$ transmits like H_{iy}.

This completes the derivation of the integral representation of the electric field everywhere in the stratified medium. These integrals are known as ***Sommerfeld integrals***. The case when the source is embedded in a layered medium can be derived similarly.

(b) Horizontal Electric Dipole (HED)—Spectral Expansions

The HED is more complicated. Unlike the VED that excites only the TM-to-z waves, an HED will excite both TE-to-z and TM-to-z waves. For a horizontal electric dipole pointing in the x direction, $\hat{\alpha} = \hat{x}$; hence, (35.2.1) and (35.2.2) give the TM-to-z and the TE-to-z components, in anticipation of their plane wave expansions, as

$$E_z = \frac{iI\ell}{4\pi\omega\epsilon}\frac{\partial^2}{\partial z \partial x}\frac{e^{ikr}}{r}, \tag{35.2.8}$$

$$H_z = -\frac{I\ell}{4\pi}\frac{\partial}{\partial y}\frac{e^{ikr}}{r}. \tag{35.2.9}$$

Then, with the Sommerfeld identity (35.1.16), we can expand the above as

$$E_z = \pm\frac{iI\ell}{4\pi\omega\epsilon}\cos\phi \int_0^\infty dk_\rho\, k_\rho^2 J_1(k_\rho\rho)e^{ik_z|z|} \tag{35.2.10}$$

$$H_z = i\frac{I\ell}{4\pi}\sin\phi \int_0^\infty dk_\rho\, \frac{k_\rho^2}{k_z} J_1(k_\rho\rho)e^{ik_z|z|}. \tag{35.2.11}$$

Now, Equation (35.2.10) represents the wave expansion of the TM-to-z field, while (35.2.11) represents the wave expansion of the TE-to-z field in terms of Sommerfeld integrals which are plane-wave expansions in disguise. Observe that because E_z is odd about $z = 0$ in (35.2.10), the downgoing wave has an opposite sign from the upgoing wave. At this point, the above are just the primary field generated by the source.

On top of a stratified medium, the downgoing wave is reflected accordingly, depending on its wave type. Consequently, we have

$$E_{1z} = \frac{iI\ell}{4\pi\omega\epsilon_1}\cos\phi \int_0^\infty dk_\rho\, k_\rho^2 J_1(k_\rho\rho)\left[\pm e^{ik_{1z}|z|} - \tilde{R}_{12}^{TM} e^{ik_{1z}(z+2d_1)}\right], \tag{35.2.12}$$

$$H_{1z} = \frac{iI\ell}{4\pi}\sin\phi \int_0^\infty dk_\rho\, \frac{k_\rho^2}{k_{1z}} J_1(k_\rho\rho)\left[e^{ik_{1z}|z|} + \tilde{R}_{12}^{TE} e^{ik_{1z}(z+2d_1)}\right]. \tag{35.2.13}$$

Notice that the negative sign in front of $\tilde{R}_{12}^{TM}$ in (35.2.12) follows because the downgoing wave in the primary field has a negative sign as shown in (35.2.10).

35.3 Stationary Phase Method—Fermat's Principle

Sommerfeld integrals are rather complex, and by themselves, they do not offer much physical insight into the physics of the field. To elucidate the physics, we can apply the stationary phase method to find approximations of these integrals when the frequency is high, or kr is large, or the observation point is many wavelengths away from the source point. It turns out that this method is initmately related to Fermat's principle.

In order to avoid having to work with special functions like Bessel functions, we convert the Sommerfeld integrals back to spectral integrals in the cartesian coordinates. We could have obtained the aforementioned integrals in cartesian coordinates were we to start with the Weyl identity instead of the Sommerfeld identity. To do the back conversion, we make use of the identity,

$$\frac{e^{ik_0 r}}{r} = \frac{i}{2\pi} \iint\limits_{-\infty}^{\infty} dk_x dk_y \frac{e^{ik_x x + ik_y y + ik_z |z|}}{k_z} = i \int\limits_{0}^{\infty} dk_\rho \frac{k_\rho}{k_z} J_0(k_\rho \rho) e^{ik_z |z|}. \tag{35.3.1}$$

We can just focus our attention on the reflected wave term in (35.2.6) and rewrite it in cartesian coordinates to get

$$\begin{aligned} E_{1z}^R &= \frac{-I\ell}{8\pi^2 \omega \epsilon_1} \iint\limits_{-\infty}^{\infty} dk_x dk_y \frac{k_x^2 + k_y^2}{k_{1z}} R_{12}^{TM} e^{ik_x x + ik_y y + ik_{1z}(z+2d_1)} \\ &= \iint\limits_{-\infty}^{\infty} dk_x dk_y \frac{1}{k_{1z}} F(k_x, k_y) e^{ik_x x + ik_y y + ik_{1z}(z+2d_1)} \end{aligned} \tag{35.3.2}$$

where we have put all the complicated terms of the integrand in the function $F(k_x, k_y)$ defined as

$$F(k_x, k_y) = \frac{-I\ell}{8\pi^2 \omega \epsilon_1} (k_x^2 + k_y^2) R_{12}^{TM}$$

In the above, $k_x^2 + k_y^2 + k_{1z}^2 = k_1^2$ is the dispersion relation satisfied by the plane wave in region 1. Also, R_{12}^{TM} is dependent on $k_{iz} = \sqrt{k_i^2 - k_x^2 - k_y^2}$ in cartesian coordinates, where $i = 1, 2$. Now the problem reduces to finding the approximation of the following integral:

$$E_{1z}^R = \iint\limits_{-\infty}^{\infty} dk_x dk_y \frac{1}{k_{1z}} F(k_x, k_y) e^{irh(k_x, k_y)} \tag{35.3.3}$$

where

$$rh(k_x, k_y) = r\left(k_x \frac{x}{r} + k_y \frac{y}{r} + k_{1z} \frac{z}{r}\right), \tag{35.3.4}$$

We want to approximate the above integral when $rh(k_x, k_y)$ is large. This happens when x, y, and z are large compared to wavelength. For simplicity, we have set $d_1 = 0$ to begin with.

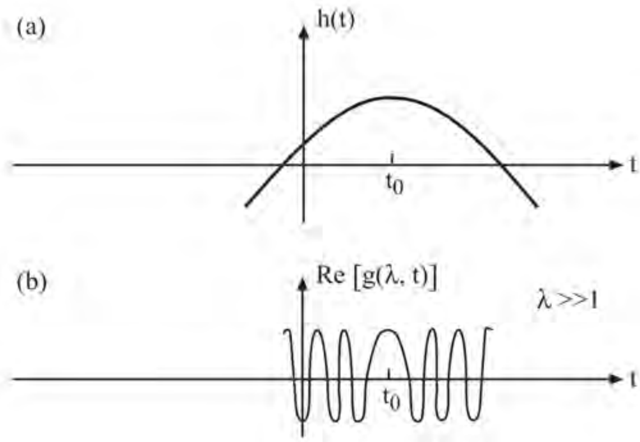

Figure 35.6: In this figure, t can represent k_x or k_y when one of them is varying. Around the stationary phase point, the function $h(t)$ is slowly varying. In this figure, $\lambda = r$, and $g(k_x, k_y, \lambda) = e^{i\lambda h(k_x,k_y)} = e^{irh(k_x,k_y)}$. When $\lambda = r$ is large, the function $g(\lambda, k_x, k_y)$ is rapidly varying with respect to either k_x or k_y. Hence, most of the contributions to the integral comes from around the stationary phase point.

In the above, $e^{irh(k_x,k_y)}$ is a rapidly varying function of k_x and k_y when x, y, and z are large, or r is large compared to wavelength.[14] In other words, a small change in k_x or k_y will cause a large change in the phase of the integrand, or the integrand will be a rapidly varying function of k_x and k_y. Due to the cancellation of the integral when one integrates a rapidly varying function, most of the contributions to the integral will come from around the stationary point of $h(k_x, k_y)$ or where the function is least slowly varying. Otherwise, the integrand is rapidly varying away from this point, and the integration contributions will destructively cancel with each other, while around the stationary point, they will add constructively.

The stationary point in the k_x and k_y plane is found by setting the derivatives of $h(k_x, k_y)$ with respect to to k_x and k_y to zero. By so doing

$$\frac{\partial h}{\partial k_x} = \frac{x}{r} - \frac{k_x}{k_{1z}}\frac{z}{r} = 0, \qquad \frac{\partial h}{\partial k_y} = \frac{y}{r} - \frac{k_y}{k_{1z}}\frac{z}{r} = 0 \tag{35.3.5}$$

The above represents two equations from which the two unknowns, k_{xs} and k_{ys}, at the stationary phase point can be solved for. By expressing the above in spherical coordinates, $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, the values of (k_{xs}, k_{ys}), that satisfy the above equations are

$$k_{xs} = k_1 \sin\theta\cos\phi, \quad k_{ys} = k_1 \sin\theta \sin\phi \tag{35.3.6}$$

with the corresponding $k_{1zs} = k_1 \cos\theta$.

[14]The yardstick in wave physics is always wavelength. Large distance is also synonymous to increasing the frequency or reducing the wavelength.

When one integrates on the k_x and k_y plane, the dominant contribution to the integral will come from the point in the vicinity of (k_{xs}, k_{ys}). Assuming that $F(k_x, k_y)$ is slowly varying, we can equate $F(k_x, k_y)$ to a constant equal to its value at the stationary phase point, and say that

$$E_{1z}^R \simeq F(k_{xs}, k_{ys}) \iint_{-\infty}^{\infty} \frac{1}{k_{1z}} e^{ik_x x + ik_y y + ik_{1z} z} dk_x dk_y = 2\pi F(k_{xs}, k_{ys}) \frac{e^{ik_1 r}}{ir} \tag{35.3.7}$$

In the above, the integral can be performed in closed form using the Weyl identity.

The above expression has two important physical interpretations.

(i) Even though a source is emanating plane waves in all directions in accordance to (35.1.12), at the observation point r far away from the source point, only one or few plane waves in the vicinity of the stationary phase point are important. They interfere with each other constructively to form a spherical wave that represents the ray connecting the source point to the observation point. Plane waves in other directions interfere with each other destructively, and are not important. That is the reason that the source point and the observation point is connected only by one ray, or one bundle of plane waves in the vicinity of the stationary phase point. These bundle of plane waves are also almost paraxial with respect to each other. This yields the insight that a ray is a bundle of plane waves who are paraxial with respect to each other.

(ii) The function $F(k_x, k_y)$ could be a very complicated function like the reflection coefficient R^{TM}, but only its value at the stationary phase point matters. If we were to make $d_1 \neq 0$ again in the above analysis, the math remains similar except that now, we replace r with $r_I = \sqrt{x^2 + y^2 + (z + 2d_1)^2}$. Due to the reflecting half-space, the source point has an image point as shown in Figure 35.7 This physical picture is shown in the figure where r_I now is the distance of the observation point to the image point. The stationary phase method extract a ray that emanates from the source point, bounces off the half-space, and the reflected ray reaches the observer modulated by the reflection coefficient R^{TM}. But the value of the reflection coefficient that matters is at the angle at which the incident ray impinges on the half-space.

(iii) At the stationary point, the ray is formed by the **k**-vector where $\mathbf{k} = \hat{x} k_1 \sin\theta \cos\phi + \hat{y} k_1 \sin\theta \sin\phi + \hat{z} k_1 \cos\theta$. This ray points in the same direction as the position vector of the observation point $\mathbf{r} = \hat{x} r \sin\theta \cos\phi + \hat{y} r \sin\theta \sin\phi + \hat{z} r \cos\theta$. In other words, the **k**-vector and the **r**-vector point in the same direction. This is reminiscent of Fermat principle, because when this happens, the ray propagates with the minimum phase between the source point and the observation point. When $z \rightarrow z + 2d_1$, the ray for the image source is altered to that shown in Figure 35.7 where the ray is minimum phase from the image source to the obervation point. Hence, the stationary phase method is initimately related to Fermat's principle.

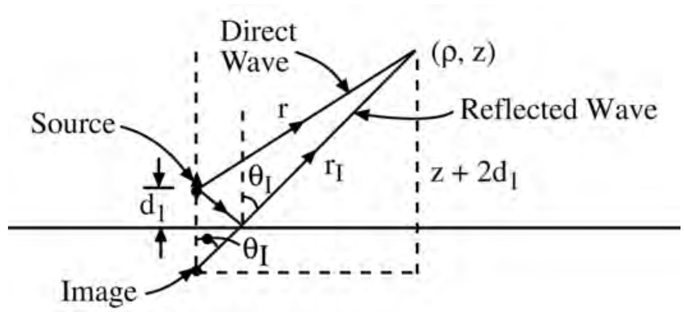

Figure 35.7: At high frequencies, the source point and the observation point are connected by a ray. The ray represents a bundle of plane waves that interfere constructively. This even true for a bundle of plane waves that reflect off an interface. So ray theory or ray optics prevails here, and the ray bounces off the interface according to the reflection coefficient of a plane wave impinging at the interface with θ_I.